Tightening Curves on Surfaces Monotonically and Applications

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Surfaces and embedded graphs

- A surface is a space that looks locally like the plane.
- In this talk we care about connected, orientable surfaces, which are classified by their number of holes (genus) and their number of boundaries.

A closed curve on a surface $S$ is a continuous map $S^1 \to S$.
- We consider closed curves in general position, e.g., with a finite number of transverse intersections.
Tightening curves

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![Diagram showing homotopy moves](image)
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Questions

1. How to compute the tightened curves? *(minimal position)*
2. How many moves are needed?
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*There exists a monotone sequence of homotopy moves to tighten any collection of closed curves.*
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Some answers

Theorem (Chang, dM ’20)

$\tilde{O}(n^5/g^2 + gn^3)$ monotone homotopy moves are sufficient to tighten any collection of curves with $n$ crossings on a surface* of genus $g$.

These moves can be computed in the same time, thus giving a polynomial-time algorithm.

Previously known:

- Exponential bound on monotone homotopy moves.
- Polynomial-time algorithm to compute minimal position of a single curve [Despré Lazarus ’17].
- Polynomial bound on homotopy moves for a single curve but not monotone [Chang Erickson Letscher dM Schleimer Sedgwick Tillmann Thurston ’18].

*: except the torus (!)
Why would I care about monotone homotopy moves for multiple curves?
Electrical moves

Take a graph $G$ embedded on a surface. Allow it to be modified with facial \textit{electrical moves}:
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These moves are useful for *resistor networks* [Kennelly 1899],
Electrical moves

Take a graph $G$ embedded on a surface. Allow it to be modified with facial electrical moves:

These moves are useful for polyhedra [Steinitz 1916],
Electrical moves

Take a graph $G$ embedded on a surface. Allow it to be modified with facial *electrical moves*:

- These moves are useful for *structural graph theory* [Robertson Seymour Thomas 1993]...
How to make a graph as small as possible with electrical moves?

Previous work: planar graphs with at most 4 terminals [Demasi Mohar 2015].

Topological approach:
- Consider the **medial graph**.
- Vertices have degree four → family of curves.
- Almost like homotopy moves!

- Diagrams of possible moves and transformations in electrical graphs.
Minimizing graphs with electrical moves

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Minimizing electrical graphs in polynomial time

One solution:
1. Use \textit{monotone} homotopy moves to tighten the curves of the medial graph,
2. but instead of $\triangleleft \rightarrow \triangleleft$, do $\triangleleft \rightarrow \times$,
3. restart the tightening with the new family of curves, loop to (1),
4. end when the family of curves that is tight.

\textbf{Complexity}

The number of crossings never increases, and, using our theorem, step (1) is polynomially long,
$\rightarrow$ polynomial number of moves.

That would not work with a non-monotone sequence of moves.

\textbf{Theorem (Chang Cossarini Erickson ’19)}

\textit{If a family of curves is tight for homotopy moves, it is also tight for electrical moves.}
Back to tightening curves monotonically
Friends...

- When tightening curves, empty monogons \( \bigcirc \) and bigons \( \bigtriangledown \) can be removed instantly.
- But what if there is stuff inside?

**Theorem (Steinitz 1916)**

*A minimal bigon always has a triangle incident to one of the two curves, and thus can be emptied and removed in a linear number of moves.*
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Theorem (Steinitz 1916)

A minimal bigon always has a triangle incident to one of the two curves, and thus can be emptied and removed in a linear number of moves.
... and foes

But when the bigon is not embedded, things get tricky:
Outside help: geometry

- A **hyperbolic metric** on a surface is a metric with constant negative curvature, i.e., points are locally isometric to saddles.

**Theorem**

*Any surface of genus* $g$ *with* $b$ *boundaries such that* $2g + b > 2$ *can be endowed with a hyperbolic metric.*

In this talk, we focus on those surfaces with $2g + b > 2$.

- A **geodesic** is a closed curve minimizing lengths locally.
Outside help

Hyperbolic geometry helps a lot because, with such a metric

- Every closed curve is homotopic to a unique geodesic, and
- primitive Geodesics are in minimal position.

Our first step is thus to endow $S$ with a (well chosen) hyperbolic metric and **straighten** the curves for that metric.

Pushing all the way to geodesics is too expensive but we can push curves to a small neighborhood of the geodesics.

- Everything outside of this small neighborhood will never be needed in the homotopy. → Surface with boundary.
To push the curves towards their geodesics, we straighten paths iteratively in *disks*.

**Theorem (Ringel 1956)**

*Paths in a Euclidean or hyperbolic disk can be straightened in a polynomial number of monotone homotopy moves.*
Straightening

Disks well chosen + hyperbolic computations
→ polynomial number of rounds to get to a small neighborhood of the geodesics that does not cover the whole surface.
→ Reduction to the case of a surface with boundary.

Caveat

Does not work for the torus.
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In hyperbolic geometry, a curve equidistant to a geodesic is \textit{not} a geodesic. This is what allows us to quantify our progress.
Disks well chosen + hyperbolic computations
\[\rightarrow\] polynomial number of rounds to get to a small neighborhood of the geodesics that does not cover the whole surface.
\[\rightarrow\] Reduction to the case of a surface with boundary.

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Surfaces with boundary

Topologically, a surface with boundary is like the neighborhood of a graph.

- Curves on the surface correspond to walks on the graph.
- We can tinker the curves so that the walks contain no spurs and all the crossings happen in the clusters (=vertices) of the graph.

These walks are topological invariants of the curves.

Main tool

Cluster and pipe expansions from graph drawing [Cortese di Battista Patrignani Pizzonia ’09]
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Topologically, a surface with boundary is like the neighborhood of a graph.

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Main tool

Cluster and pipe expansions from graph drawing [Cortese di Battista Patrignani Pizzonia ’09]
**Straighten** the curve in the clusters, and **expand** the clusters.

Morally, it is safe to do so because the walks force the situation at the cluster.
Pipe expansions

Then *expand* the *pipes* (=edges).

Some potential drops at each step...
...and after a polynomial number of rounds, the curves do not branch at the clusters anymore, i.e., look like braids.

Now, the situation is roughly the same as for curves in an annulus, for which a quadratic bound exists [Geck and Pfeiffer ’93, de Graaf and Schrijver ’97].
Concluding words

Monster in the closet: *terminal-leaf reductions*

Two open problems:

1. Solve the case of the torus.
2. Obtain tight bounds, notably:

**Feo-Provan Conjecture**

In the plane, the number of monotone homotopy moves to tighten a collection of closed curves/electrical moves to reduce a graph to a point is $\Theta(n^{3/2})$.

The non-monotone lower bound and upper bound are known [Chang Erickson ’17].
Thank you! Questions?